The Mathematics of Hyperbolic Grading

Special Relativity, Elo Ratings, and Stereographic Projection

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Abstract

This document serves as the theoretical manual for the **Hyperbolic Grading Simulator**. We propose a rigorous unification of Item Response Theory (Psychometrics) and Special Relativity. By reinterpreting a grade not as a linear score but as a normalized **expected payoff** (velocity), we demonstrate an exact isomorphism between the Elo rating system and the geometry of Minkowski spacetime. We prove that while Elo competence θ behaves like an additive rapidity on a hyperbola, the observable grade v is its stereographic projection onto a bounded segment.

1 The Linearity Problem

Current educational assessment suffers from an arithmetic incoherence. Grades are typically treated linearly ($N' = N + \delta$) on a bounded scale (e.g., $[0, 100]$). This approach fails for two reasons:

1. **Boundary Violations (Clipping):** Adding points to a score of 99/100 mathematically pushes it "outside" the scale, forcing arbitrary capping.
2. **The Weber-Fechner Law:** Improving from 50% to 60% requires significantly less effort than improving from 98% to 99%. Linear grading ignores the exponential cost of approaching perfection.

Just as a particle cannot linearly accelerate past the speed of light c, a student cannot linearly improve past 100% certainty. Therefore, grading geometry is not Euclidean; it is Lorentzian.

2 Mathematical Foundations

The core of the theory relies on a physical redefinition of the "Grade" to align it with probabilistic truth.

2.1 The Grade as Expected Payoff (v)

Consider an assessment as a Bernoulli trial with parameter p (the intrinsic probability of success).

- **Success (100%):** Payoff $+1$
- **Failure (0%):** Payoff -1 (loss)

The **Expected Payoff** v is defined as:

$$v = (+1) \cdot p + (-1) \cdot (1 - p) = 2p - 1 \tag{1}$$

This variable v is strictly bounded on $[-1,1]$. In our physical isomorphism, we identify v as a **dimensionless velocity** (where $c = 1$).

2.2 The Elo Model (θ)

In psychometrics (specifically the Rasch or Elo models), the probability of success p is determined by the difference in competence θ (the *log-odds*) between the student and the difficulty of the task:

$$p(\theta) = \frac{1}{1 + e^{-\theta}} \tag{2}$$

Here, θ extends from $-\infty$ to $+\infty$. This is the invariant quantity representing "Aptitude."

2.3 The Fundamental Isomorphism

Injecting (2) into (1) yields the constitutive relation of the theory.

Theorem 1 (The Probability-Velocity Link)**.** *The expected payoff (grade) v associated with an Elo competence θ is given by the hyperbolic tangent of the half-angle:*

$$v = \tanh\left(\frac{\theta}{2}\right) \tag{3}$$

Proof. Substituting $p(\theta)$ into the expectation formula:

$$v = 2\left(\frac{1}{1 + e^{-\theta}}\right) - 1 = \frac{2 - (1 + e^{-\theta})}{1 + e^{-\theta}} = \frac{1 - e^{-\theta}}{1 + e^{-\theta}}$$

Multiply numerator and denominator by $e^{\theta/2}$:

$$v = \frac{e^{\theta/2} - e^{-\theta/2}}{e^{\theta/2} + e^{-\theta/2}} = \frac{2\sinh(\theta/2)}{2\cosh(\theta/2)} = \tanh\left(\frac{\theta}{2}\right)$$

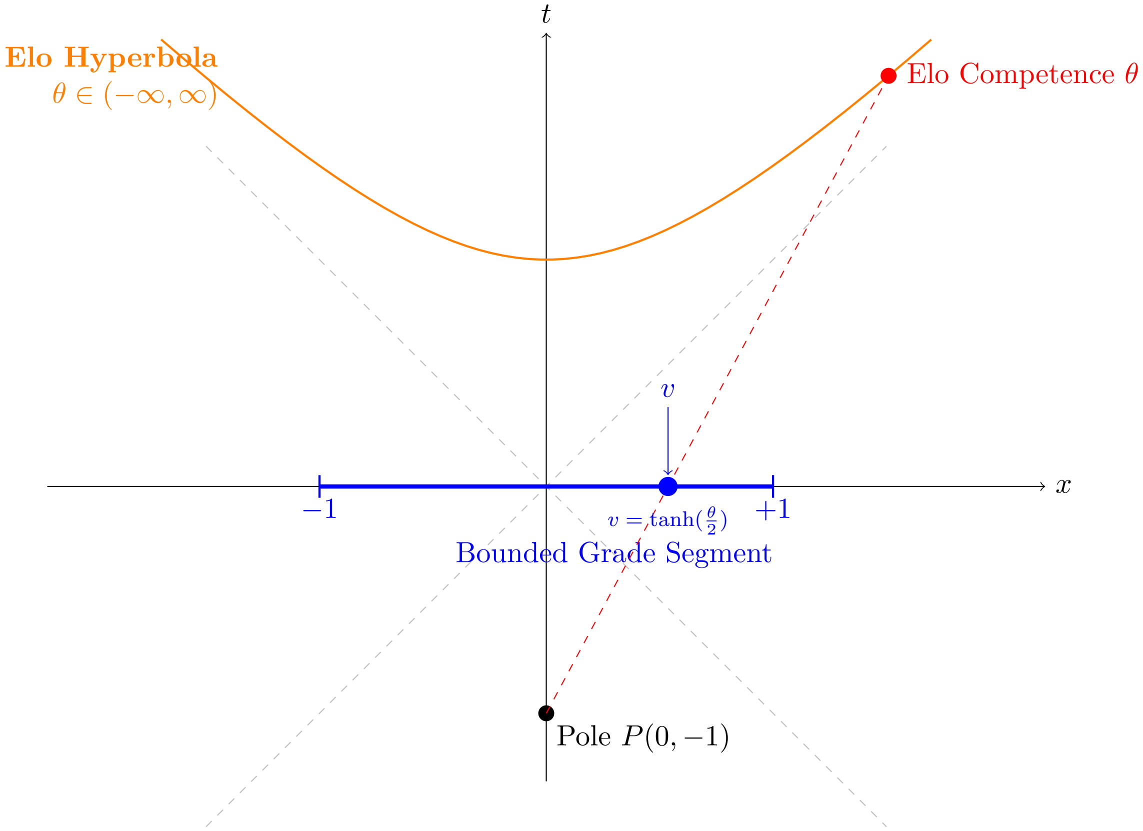
□

This equation is remarkable. In standard Special Relativity, velocity is $v = \tanh(\eta)$ where η is rapidity. In Psychometrics, the grade is $v = \tanh(\theta/2)$. This suggests that **Elo Competence is twice the physical rapidity**, a scaling factor naturally resolved by the geometry of projection.

3 Geometric Interpretation: Stereographic Projection

The simulator visualizes this relationship using a Minkowski diagram. We map the infinite "Hyperbola of Competence" onto the finite "Exam Paper" via stereographic projection.

Proposition 1. *If we identify Elo competence θ with the position on the unit hyperbola $t^2 - x^2 = 1$, then the grade v is exactly the stereographic projection of this state onto the axis $t = 0$ from the pole $P(0, -1)$.*



Geometric Proof: Let $S(\sinh\theta, \cosh\theta)$ be a point on the hyperbola representing the student's Elo rating. The line (PS) connecting the pole $P(0,-1)$ to S intersects the horizontal axis ($t = 0$) at abscissa x. The slope of the line (PS) is:

$$m = \frac{\cosh\theta - (-1)}{\sinh\theta - 0} = \frac{\cosh\theta + 1}{\sinh\theta}$$

The equation of the line is $t = m \cdot x - 1$. Setting $t = 0$ to find the intersection:

$$0 = m \cdot x - 1 \implies x = \frac{1}{m} = \frac{\sinh\theta}{\cosh\theta + 1}$$

Using half-angle hyperbolic identities ($\sinh\theta = 2\sinh\frac{\theta}{2}\cosh\frac{\theta}{2}$ and $\cosh\theta + 1 = 2\cosh^2\frac{\theta}{2}$):

$$x = \frac{2\sinh(\frac{\theta}{2})\cosh(\frac{\theta}{2})}{2\cosh^2(\frac{\theta}{2})} = \tanh\left(\frac{\theta}{2}\right)$$

This confirms that the stereographic projection of the Elo parameter θ perfectly recovers the expected payoff v.

4 Relativistic Adjustment (Boost)

In the simulator, changing the "Difficulty" of the exam corresponds to changing the inertial reference frame.

If an exam has a difficulty calibration δ (also an angle on the hyperbola), the effective grade v' is not found by linear subtraction $(\theta - \delta)$. Instead, we must use the **Velocity Addition Law** for the projected grades.

Let $v = \tanh(\theta/2)$ be the student's raw grade and $u = \tanh(\delta/2)$ be the difficulty's "velocity." The observed grade v_{obs} is:

$$v_{obs} = \frac{v - u}{1 - v \cdot u} \tag{4}$$

Insight 1. *This formula guarantees that v_{obs} never exceeds ± 1. It visually explains why a "hard" exam (high u) compresses the grades of average students towards -1 (failure), while only slightly lowering the grades of top students (time dilation effect).*

5 Conclusion

The Hyperbolic Grading Simulator demonstrates that:

1. **Grades are Projections:** They are distorted, finite shadows of an unbounded aptitude.
2. **Aptitude is Invariant:** The Elo θ exists independently of the test.
3. **Difficulty is Relative:** It is merely the coordinate choice of the observer.

Ultimately, this framework offers a rigorous path for **harmonizing assessment standards**. By explicitly calibrating the "difficulty velocity" of exams, educational institutions could mathematically translate grades across different schools or national systems. Just as relativity allows physicists to reconcile observations from different reference frames, Hyperbolic Grading paves the way for a **Covariant Assessment System**—one where student aptitude can be universally compared, regardless of the local difficulty of the tests they faced.